Integer solutions using coefficient method

Suppose there are 20 intermediary stations between two stations A and B. A train can stop at 3 of these stations but there must be minimum 3 stop between any two consecutive stoppings. In how many ways the train can reach its destination.

To solve questions of these type, we should learn a important method called coefficient method.

The number of non - negative intezer solutions for the equation $x_1+x_2+x_3+...x_r=n$ is the number of ways of distribution n identical things among r persons. , when each person can get zero, one or more things.

This is nothing but Coefficient of x^n in $[1+x+x^2+......x^n]^r$

The terms in the bracket are in geometric progression with common ratio of x. And they are (n+1) terms. Applying the geometric progression sum rule

= Coefficient of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$

= Coefficient of x^n in $(1-x^{n+1})^r(1-x)^{-r}$

In the first expression, $[1+{}^rC_1x^{n+1}+{}^rC_2(x^{n+1})^2+...](1-x)^{-r}$ $=(1-x)^{-r}$

[first term in the first expression only gives a power of n, all other terms have powers are in the multiples of n+1]

= Coefficient of x^n in $(1+x+x^2+x^3+.......)$

= ${}^{(n+r-1)}C_{r-1}$

How to apply the above formula:

$(1-x)^{-18}=1+{}^{18}C_1x+{}^{19}C_2x^2+{}^{20}C_3x^3+.......\infty$

The coefficient of x^3 will be calculated by using the formula = ${}^{(n+r-1)}C_{r-1}$ = ${}^{(3+18-1)}C_{18-1}={}^{20}C_{17}={}^{20}C_3$

1. Find the number of positive number of solutions of x + y + z + w = 20 (a) if "0" is allowed (b) if "0" is not allowed.

This equation is nothing but distributing 20 similar articles to 4 persons. Here n = 20 and r = 4

Applying the formula ${}^{(n+r-1)}C_{r-1}={}^{20+4-1}C_{4-1}={}^{23}C_3=1771$.

If 0 is not allowed, the let us give one article each to x, y , z , w. Now assume $x=x^1+1$, $y=y^1+1$, $z=z^1+1$, $w=w^1+1$

So our equation becomes, $(x^1+1)+(y^1+1)+(z^1+1)+(w^1+1)=20$

Now x^1,y^1,z^1,w^1 can take zero.

So $x^1+y^1+z^1+w^1=16$ and number of intezer solutions for this equation become

${}^{(n+r-1)}C_{r-1}={}^{16+4-1}C_{4-1}={}^{19}C_3=969$

Note: When we give r articles each one to r persons we left with (n-r) articles. Now distributions these articles to r persons = $^{(n-r)+r-1}C_{r-1} = {}^{n-1}C_{r-1}$

If x, y, z , w are graeter than equal to 1, we apply above formula.

2. How many integral solutions exist for x + y + z + w = 29 where $x \geq 1, y \geq 1, z \geq 3$ and $w \geq 0$

Sol: x + y + z + w = 29

Take $x = x^1 + 1$, $y = y^1 + 1$, $z = z^1 + 3$

Substituting these values in our equation makes it as $(x^1 + 1) + (y^1 + 1) + (z^1 + 3) + w = 29$

$x^1 + y^1 + z^1 + w = 23$

Number of intezer solutions for the above equation = $^{(n+r-1)}C_{r-1} = {}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$

3. How many integral solutions are there to the system of equations a+b+c+d+e = 20 and a + b = 15 where "0" is allowed.

Sol: a+b+c+d+e = 20 -----(1)

a + b = 15 ----(2)

From the above two equations, c+d+e = 5 -----(3)

Number of intezer solutions for (2) is $^{15+2-1}C_{2-1} = {}^{16}C_1 = 16$ and for (3) is $^{5+3-1}C_{3-1} = {}^{7}C_2 = 21$

So total solutions are 16 x 21 = 336

4. Find the number of intezer solutions for the equation x + y + z + 4t = 20

Sol: The values for 4t are 0, 4, 8, 12, 16, 20

By substituting the above values in the given equation, we get x + y + z = 20, x + y + z = 16, x + y + z = 12, x + y + z = 8, x + y + z = 4, x + y + z = 0

Now the solutions for the above equations are $^{20+3-1}C_{3-1} = {}^{22}C_2$, $^{16+3-1}C_{3-1} = {}^{18}C_2$, $^{12+3-1}C_{3-1} = {}^{14}C_2$, $^{8+3-1}C_{3-1} = {}^{10}C_2$, $^{4+3-1}C_{3-1} = {}^{6}C_2$, and 1

So total solutions = $^{22}C_2, {}^{18}C_2, {}^{14}C_2, {}^{10}C_2, {}^{6}C_2$, 1 = 231 + 153 + 91 + 45 + 15 + 1 = 536

5. There are 20 intermediary stations between two stations A and B. A train can stop at 3 of these stations but there must be minimum 3 stop between those intermediary stoppings. In how many ways the train can reach its destination.

Sol: $A, S_1, S_2 \ldots\ldots S_K, \ldots\ldots\ldots S_L, \ldots\ldots\ldots\ldots S_M, \ldots\ldots\ldots\ldots S_{20}, B$

Assume that there are P stations between A and S_K , Q stations between S_K and S_L , R stations between S_L and S_M and S stations between S_M and B.

Now there must be 3 stoppings between S_K, S_L, S_M but there need not be any stopping between A and the first intermedIry and B and the last intermediary stations.

This gives us, P + Q + R + S = 17 (As we have to substract 3 stations from the sum of intermediary stations)

Here P $\geq$ 0, Q $\geq$ 3, R $\geq$ 3 and S $\geq$ 0

Substituting $Q = Q^1 + 3$ and $R = R^1 + 3$ we get $P + Q^1 + R^1 + S = 11$

Now number of intezer solutions for this equation = $^{11+4-1}C_{4-1} = {}^{14}C_3$

Integer solutions when an integer has some minimum and maximum limit:

We have seen already that, how to find integer solutions x + y + z + w = 20 where x, y, z, w may take values from 0 to 20. But what if when x, y, z, w has a minimum of 3 and maximum limit of 10. ie., We may not substitute values more than 10 in this equation. So how do we attempt to solve this equation?

Suppose, we have n similar articles may be distributed to 1 person. In how many ways we can distribute 3 articles to this person? In how many ways we can distribute 5 articles to this person? In how many ways n articles to be distributed to this person?

For all above questions, the answer is 1 as all articles are similar there is only 1 ways to choose, 2, 5, n articles from n articles.

So number of ways of distributing K articles to 1 person is the cofficient of $x^k = 1+x+x^2+x^3+......x^n$ which is 1.

Now assume He will get a minimum of 3 and maximum of 6, then we have to consider this equation.

$x^3+x^4+x^5+x^6$

6. How many integers between 1 to 1000000 have the sum of the digits 18?

Sol: Any number between 1 to 10000000 has 7 digits. Let us say they are $a_1, a_2, a_3.....a_7$. Now

$a_1+a_2+a_3+a_4+a_5+a_6+a_7=18$

Here these variables take a minimum value of 0 but a maximum value of 9. 18, 0, 0, 0, 0, 0, 0 is a solution of the above but this is violating our condition as all the digits are single digit numbers of maximum 9

We have to consider, x^{18} coefficient in the expansion of $(x^0+x^1+x^2+x^3+x^4......x^9)^6$

The required number is x^{18} coefficient in the expansion of $\left(\frac{1-x^{10}}{1-x}\right)^6$

x^{18} coefficient in the expansion of $(1-x^{10})^6(1-x)^{-6}$

x^{18} coefficient in the expansion of $(1-{}^6C_1x^{10}+...)(1-x)^{-6}$

x^{18} coefficient in the second expression multiplied by 1 will give us one x^{18} and x^8 coefficient in the second expression multiplied by x^{10} will give us another x^{18} coefficient.

= Coefficient of x^{18} in $(1-x)^{-6}$ - ${}^6C_1$. Coefficient of x^8 in $(1-x)^{-6}$

= ${}^{6+18-1}C_{6-1}-{}^6C_1.{}^{6+8-1}C_{6-1} = {}^{23}C_5-6.{}^{13}C_5=25927$

7. In how many ways can we get a sum of 12 throwing 3 dice.

Sol: A single dice shows from 1 to 6. So we have to find the integer solutions for A + B + C = 12 where A, B, C represent the numbers of dice subject to the condition that they take values form 1 to 6.

So we have to find coefficient of x^{12} in the expansion $(x^1+x^2+x^3+x^4+x^5+x^6)^3$

= Coefficient of x^{12} in $x^3(1+x+x^2+x^3+x^4+x^5)^3$

= Coefficient of x^9 in $\left(\frac{1-x^6}{1-x}\right)^3$

= Coefficient of x^9 in $(1-x^6)^3(1-x)^{-3}$

= Coefficient of x^9 in $(1-{}^3C_1x^6+{}^3C_2x^{12}...)(1-x)^{-3}$

Required coefficient is coefficient of x^9 in $(1-x)^{-3}$ - ${}^3C_1$. Coefficient of x^3 in $(1-x)^{-3}$

= ${}^{9+3-1}C_{3-1}-3.{}^{3+3-1}C_{3-1} = {}^{11}C_2-3.{}^5C_2=25$

8. Find the number of non negitive integer solutions of the inequality x + y + z $\leq$ 20

Sol: We add another variable K to this equation to make it an equality

Now x + y + z + K = 20 where K can take any value from 0 to 20 so that we get different solutions for the above equation from 20 to 0.

Now number of solutions for the above equation = $^{(n+r-1)}C_{r-1} = {}^{20+4-1}C_{4-1} = {}^{23}C_3$

9. If 3 dice are rolled, The number of ways so that the sum of the numbers on them is atleast 9 is

Sol: We first consider that the maximum sum on the dice is 8.

Now A + B + C $\leq$ 8

Add a variable K to this equation to make it an equality

A + B + C + K = 8 Subjected to the condition A, B, C takes values from 1 to 6 and K take any value from 0 to 8 for the sum of the numbers ranges from 8 to 0

We have to find the coefficient of x^8 in